THE 3-D INVISCID LIMIT RESULT UNDER SLIP BOUNDARY CONDITIONS. A NEGATIVE ANSWER

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Abstract

We show that, *in general*, the solutions to the initial-boundary value problem for the Navier-Stokes equations under a widely adopted Navier-type slip boundary condition do not converge, as the viscosity goes to zero (in any arbitrarily small neighborhood of the initial time), to the solution of the Euler equations under the classical zero-flux boundary condition, and same smooth initial data. Convergence does not hold with respect to any space-topology which is sufficiently strong as to imply that the solution to the Euler equations inherits the complete slip type boundary condition (see the Theorem 1.2 below). In our counter-example Ω is a sphere, and the initial data may be infinitely differentiable. The crucial point here is that the boundary is not flat. In fact (see [3]), if $\Omega = \mathbb{R}^3_+$, convergence holds in $C([0,T]; W^{k,p}(\mathbb{R}^3_+))$, for arbitrarily large k and p. For this reason, the negative answer given here was not expected.

1 Introduction

In some recent papers, see [1], [2], [3], we have considered the problem of the strong convergence up to the boundary, as $\nu \to 0$, of the solutions u^ν of the Navier-Stokes equations in the cylinder $\Omega \times (0,T)$

$$\left\{ \begin{array}{l} \partial_t \underline{u}^\nu + (\underline{u}^\nu \cdot \nabla)\, \underline{u}^\nu - \nu\, \Delta\, \underline{u}^\nu + \nabla\, \pi = 0, \\ div\, \underline{u}^\nu = 0\,, \\ \underline{u}^\nu(0) = \underline{u}_0\,, \end{array} \right. \tag{1.1}$$

under the slip boundary conditions at $\partial\Omega \times (0,T)$

$$\left\{ \begin{array}{l} \underline{u}^\nu \cdot \underline{n} = 0, \\ \underline{\omega}^\nu \times \underline{n} = 0\,, \end{array} \right. \tag{1.2}$$

where $\underline{\omega} = curl\, \underline{u}$, to the solution u of the Euler equations

$$\left\{ \begin{array}{l} \partial_t \underline{u} + (\underline{u} \cdot \nabla)\, \underline{u} + \nabla\, \pi = 0, \\ div\, \underline{u} = 0\,, \\ \underline{u}(0) = \underline{u}_0\,, \end{array} \right. \tag{1.3}$$

under the zero flux boundary condition

$$\underline{u} \cdot \underline{n} = 0\,. \tag{1.4}$$

The domain Ω is an open set in $\mathbb{R}^3$ locally situated on one side of its boundary Γ, and $\underline{n} = (n_1, n_2, n_3)$ is the unit outward normal to Γ. We have showed

that strong convergence holds provided that the boundary is flat. In particular, in the half-space case we proved [3] that if the initial data are in $W^{k,p}(\mathbb{R}^3_+)$, then convergence holds in $C([0,T];W^{k,p}(\mathbb{R}^3_+))$, for arbitrarily large k and p. Moreover, a minimal set of independent, necessary and sufficient, compatibility conditions on Γ at $t=0$ is displayed. These conditions appear only if $k \geq 4$.

The natural next step is to study if and how the above results continue to hold in the presence of non-flat boundaries. As a matter of fact, in the two-dimensional case the answer turns out to be positive; see, for instance, [5]. In the three dimensional case, the strong inviscid limit appears, instead, to be a much more complicated issue and, so far, an open problem; see [1] for a quite complete discussion on this problem, and for proofs of related useful equations.

In the recent paper [7] an interesting new approach to the problem is introduced. Notwithstanding, the method of proof only fully works if the boundary is flat. This fact was pointed out in the subsequent papers [1] and [2] where it was emphasized that the non-flat boundary problem remains still unsettled; for a review on these results see also [6].

Objective of this note is to show that a strong inviscid limit result, in the presence of non-flat boundaries, is false in general. Roughly speaking by "strong" we mean that it is taken in function spaces such that all the derivatives that appear in the equations, including the boundary conditions, are integrable. In particular the result is false in general, when Ω is the unit sphere, and for $C^\infty(\overline{\Omega})$ divergence free initial data which satisfies the slip boundary conditions (1.2). For instance, as ν tends to zero, the solutions to the Navier-Stokes equations do not converge to the solution of the Euler equations in $L^1(0,t_0;W^{s,q})$, for any arbitrarily small $t_0 > 0$, any $q \geq 1$, and any $s > 1 + \frac{1}{q}$. Note that the above unique solution to the Euler equations is infinitely differentiable, and the above solutions to the Navier-Stokes equations are "smooth".

A $W^{2,p}$ vanishing viscosity limit result in general domains was recently claimed in [4], Theorem 1.1. In the first part of this preprint, the authors review methods and arguments previously introduced and developed in references [1] and [2]. After this review, the authors go to the proof of the main result, their Theorem 1.1. In doing this, they partially appeal to some general ideas developed in a sequence of papers by one of us, introduced to study sharp singular limit problems. In fact, this approach in the present context seems to us a good choice. Actually, the layout of the paper is convincing. Unfortunately, the final result is incompatible with the counter-example presented below.

Remark 1.1. On flat portions of the boundary, the slip boundary conditions coincide with the classical Navier boundary conditions

$$\begin{cases} \underline{u}\cdot\underline{n} = 0, \\ \underline{t}\cdot\underline{\tau} = 0, \end{cases} \tag{1.5}$$

where $\underline{\tau}$ stands for any arbitrary unit tangential vector. Here $\underline{t}$ is the stress vector defined by $\underline{t} = \mathcal{T}\cdot\underline{n}$, where the stress tensor $\mathcal{T}$ is defined by

$$\mathcal{T} = -\pi\, I + \frac{\nu}{2}\,(\nabla\underline{u} + \nabla\underline{u}^T)\,.$$

These conditions were introduced by Navier in 1823 and derived by Maxwell in

1879 from the kinetic theory of gases. In the general case

$$t \cdot \underline{\tau} = \frac{\nu}{2}\,(\underline{\omega} \times \underline{n}) \cdot \underline{\tau} - \nu\,\mathcal{K}_\tau\,\underline{u} \cdot \underline{\tau}\,, \tag{1.6}$$

where $\mathcal{K}_\tau$ is the principal curvature in the $\underline{\tau}$ direction, positive if the corresponding center of curvature lies inside Ω.

Note that our counter-example does not exclude that strong vanishing results hold under the Navier boundary conditions in the non-flat boundary case.

We end the introduction by stating the following two theorems.

Theorem 1.1. *Let $\Omega = \{x : |\,x\,| < 1\}$, be the 3-dimensional unitary sphere. There is an explicit family (see the Theorem 3.1) of $C^\infty(\overline{\Omega})$, divergence free initial data $\underline{u}_0$, which satisfies the slip boundary conditions* (1.2)*, and such that the following holds. Given an element $\underline{u}_0$ belonging to the above family, there exists a $t_0 > 0$ such that the corresponding (unique, indefinitely differentiable) local solution $\underline{u}(t)$ to the Euler equations* (1.3)*,* (1.4) *does not satisfy the boundary condition $\underline{\omega} \times \underline{n} = 0$, for any $t \in (0,\, t_0]$.*

In particular, the following result holds.

Theorem 1.2. *Let $\underline{u}_0$ be a given, fixed, initial data belonging to the class referred in the above theorem 1.1. Denote by $\underline{u}^\nu$ the $\nu-$family of solutions to the Navier-Stokes equations* (1.1)*,* (1.2) *with initial data $\underline{u}_0$, and denote by $\underline{u}$ the solution of the Euler equations* (1.3)*,* (1.4) *with initial data $\underline{u}_0$.*

There does not exist a $t_0 > 0$ and exponents $q \geq 1$ and $s > 1 + \frac{1}{q}$ such that $\underline{u}^\nu$ converges to $\underline{u}$ in $L^1(0,\, t_0; W^{s,\,q}(\Omega))$. The particular case $L^1(0, t_0;\, W^{2,1}(\Omega))$ is also included in this statement.

Remark 1.2. Actually the convergence in the above theorem 1.2 fails for any arbitrary subsequence, even under weaker convergence hypotheses.

Plan of the paper: In section 2 we show how to turn the proofs of the above two theorems into the construction of a suitable class of vector fields (called here "counter-examples"). In section 3 we explicitly construct the above vector fields.

2 Reduction to a functional problem in space variables

In spite of the exceptionally strong convergence results in the case of flat boundaries, at a certain point we became inclined to believe that a strong inviscid limit result is false in general. This guess led us to look for a counter-example, by reductio ad absurdum, as follows. Let $\underline{u}_0$ be a smooth divergence free initial data, which satisfies the slip boundary conditions (1.2), and denote by $\underline{u}^\nu$ and $\underline{u}$ the corresponding solutions to the above Navier-Stokes and Euler boundary value problems. Moreover, assume (per absurdum) that $\underline{u}^\nu$ converges to $\underline{u}$ as ν goes to zero, with respect to a specific $\tau-$topology, which (by assumption) is sufficiently strong as to imply that the limit $\underline{u}(t)$ inherits the boundary condition $\underline{\omega}^\nu \times \underline{n} = 0$ near $t = 0$ (for instance, convergence in $L^1(0, t_0; W^{2,1})$).

This would imply that the Euler equations (1.3) under the classical boundary condition (1.4) necessarily enjoy the following *persistency property*: if a smooth initial data satisfies the additional boundary condition $\underline{\omega}(0) \times \underline{n} = 0$, then at least for small times, $\underline{\omega}(t)$ must verify this same property (we note that this was also considered as an open problem). It follows that, in order to contradict the possibility of the above $\tau-$convergence result, it is sufficient to contradict the above persistency property for the Euler equations. Next, by arguing as follows, we turn the proof of the absence of the above persistency property into a problem concerning only the space variables. External multiplication of the Euler vorticity equation by the normal $\underline{n}$, point-wise on Γ, leads to the equation

$$\partial_t\,(\underline{\omega} \times \underline{n}) - \mathit{curl}\,(\underline{u} \times \underline{\omega}) \times \underline{n} = 0\,. \tag{2.1}$$

If the persistency property holds, the first term in the above equation must vanish identically on Γ, at time $t = 0$. Hence the second term must verify the same property, say

$$\mathit{curl}\,(\underline{u}_0 \times \underline{\omega}_0) \times \underline{n} = 0 \tag{2.2}$$

on Γ.

Consequently, in order to prove that the above persistence property does not hold and, a fortiori, that the above $\tau-$inviscid limit result does not hold in general, it is sufficient to solve the following problem.

Problem 2.1. *To exhibit a smooth, divergence free vector field $\underline{u}_0$, in a bounded, regular, open set Ω, which satisfies the slip boundary conditions everywhere on Γ, but does not satisfy, somewhere on Γ, the boundary condition* (2.2).

Below, we succeed in constructing, *globally* in Ω, a wide class of $C^\infty(\overline{\Omega})$ vector fields for which the above, negative, result holds. We assume Ω to be the 3-dimensional unitary sphere and display our vector field in spherical coordinates. Once the vector fields are known, the verification of the desired properties is straightforward.

3 The counter-example

In what follows we use spherical coordinates (r, θ, φ). For any vector field $\underline{u}$, we denote by u_r, u_θ and u_φ the components of $\underline{u}$ in the orthonormal, positively oriented, local basis $\left(\underline{e}_r, \underline{e}_\theta, \underline{e}_\varphi\right)$. Just for convenience, let us recall the expressions of $\nabla \cdot \underline{u}$ and $\underline{\omega}$ in this curvilinear coordinate system:

$$\nabla \cdot \underline{u} = \frac{1}{r^2}\frac{\partial}{\partial r}(r^2\, u_r) + \frac{1}{r\,\sin\theta}\frac{\partial}{\partial \theta}(u_\theta\,\sin\theta) + \frac{1}{r\,\sin\theta}\frac{\partial u_\varphi}{\partial \varphi}; \tag{3.1}$$

$$\begin{aligned} \mathit{curl}\,\underline{u} = {} & \frac{1}{r\sin\theta}\left(\frac{\partial}{\partial\theta}(u_\varphi\,\sin\theta) - \frac{\partial u_\theta}{\partial\varphi}\right)\underline{e}_r \\ & + \frac{1}{r}\left(\frac{1}{\sin\theta}\frac{\partial u_r}{\partial\varphi} - \frac{\partial}{\partial r}(r\,u_\varphi)\right)\underline{e}_\theta + \frac{1}{r}\left(\frac{\partial}{\partial r}(r\,u_\theta) - \frac{\partial u_r}{\partial\theta}\right)\underline{e}_\varphi\,. \end{aligned} \tag{3.2}$$

We also recall that, for a scalar field $f = f(r, \theta, \varphi)$,

$$\nabla f = \frac{\partial f}{\partial r}\underline{e}_r + \frac{1}{r}\frac{\partial f}{\partial\theta}\underline{e}_\theta + \frac{1}{r\,\sin\theta}\frac{\partial f}{\partial\varphi}\underline{e}_\varphi\,. \tag{3.3}$$

We consider the 3-dimensional unitary sphere $\Omega = \{x : r < 1\}$, and denote by Γ its boundary. The unit external normal is denoted by $\underline{n}$. Clearly $\underline{n} = \underline{e}_r$ on Γ.

Let $h(r)$ be a $C^\infty([0, +\infty))$ real function, and $g(\theta, \varphi)$ be a $C^\infty([0, \pi] \times \mathbb{R})$ real function, 2π-periodic on φ. Just for convenience, we assume that $h(r)$ vanishes in a neighborhood of $r = 0$ and $g(\theta, \varphi)$ vanishes for θ in a neighborhood of $\theta = 0$ and $\theta = \pi$ (and arbitrary φ). Set

$$G(\theta, \varphi) = \frac{\partial}{\partial \theta}\left(\sin\theta \frac{\partial g}{\partial \theta}\right) + \frac{1}{\sin\theta}\frac{\partial^2 g}{\partial \varphi^2}.$$

Theorem 3.1. *Let $\underline{u}$ be the vector field*

$$\underline{u} = -\frac{h(r)}{\sin\theta}\frac{\partial g}{\partial \varphi}\,\underline{e}_\theta + h(r)\,\frac{\partial g}{\partial \theta}\,\underline{e}_\varphi\,. \tag{3.4}$$

Then the following results hold:

i) $\nabla \cdot \underline{u} = 0$ *in* Ω, $\underline{u} \cdot \underline{n} = 0$ *on* Γ.

ii) If $h(1) + h'(1) = 0$, *then* $\underline{\omega} \times \underline{n} = 0$ *on* Γ.

iii) If $h(1) + h'(1) = 0$, *with* $h(1) \neq 0$, *and if*

$$\frac{\partial g}{\partial \varphi} \neq 0 \quad \text{and} \quad G(\theta, \varphi) \neq 0 \tag{3.5}$$

at a point P on Γ, then $[\,curl(\underline{u} \times \underline{\omega})]_\theta \neq 0$ *in a neighborhood of P. Similarly if* $h(1) + h'(1) = 0$, *with* $h(1) \neq 0$ *and if*

$$\frac{\partial g}{\partial \theta} \neq 0 \quad \text{and} \quad G(\theta, \varphi) \neq 0 \tag{3.6}$$

at a point P on Γ, then $[\,curl(\underline{u} \times \underline{\omega})\,]_\varphi \neq 0$ *in a neighborhood of P.*

Proof. Claims in $i)$ follow by a straightforward calculation, using (3.1) and recalling that $\underline{n} = \underline{e}_r$ on Γ.

By using (3.2), and by observing that (3.4) yields $u_r = \dfrac{\partial u_r}{\partial \theta} = \dfrac{\partial u_r}{\partial \varphi} = 0$ in Ω, we show that $\underline{\omega}$ is given in $\overline{\Omega}$ by

$$\begin{aligned}\underline{\omega} &= \omega_r\,\underline{e}_r + \omega_\theta\,\underline{e}_\theta + \omega_\varphi\,\underline{e}_\varphi \\ &= \frac{h(r)}{r\sin\theta}\,G(\theta,\varphi)\,\underline{e}_r - \frac{1}{r}\frac{\partial}{\partial r}(r\,h(r))\,\frac{\partial g}{\partial \theta}\,\underline{e}_\theta - \frac{1}{r\sin\theta}\frac{\partial}{\partial r}(r\,h(r))\,\frac{\partial g}{\partial \varphi}\,\underline{e}_\varphi\,.\end{aligned}$$

In particular, on Γ the vector field $\underline{\omega} \times \underline{n}$ is given by

$$\underline{\omega} \times \underline{n} = \omega_\varphi\,\underline{e}_\theta - \omega_\theta\,\underline{e}_\varphi = -\frac{1}{r\sin\theta}\frac{\partial}{\partial r}(r\,h(r))\,\frac{\partial g}{\partial \varphi}\,\underline{e}_\theta + \frac{1}{r}\frac{\partial}{\partial r}(r\,h(r))\,\frac{\partial g}{\partial \theta}\,\underline{e}_\varphi\,.$$

Therefore, if $\frac{\partial}{\partial r}(r\,h(r))|_{r=1} = 0$, we get $\underline{\omega} \times \underline{n} = 0$ on Γ. This proves ii).

Let us pass to the last point iii). From the previous steps, we have

$$u_r = \omega_\theta = \omega_\varphi = 0 \quad \text{on} \quad \Gamma\,. \tag{3.7}$$

Set $\underline{v} = \underline{u} \times \underline{\omega}$. Since $u_r = 0$ in Ω, $\underline{v}$ is given by

$$\underline{v} = (u_\theta\, \omega_\varphi \, - \, u_\varphi\, \omega_\theta)\, \underline{e}_r + u_\varphi\, \omega_r\, \underline{e}_\theta - \, u_\theta\, \omega_r\, \underline{e}_\varphi . \tag{3.8}$$

Note that $\underline{\omega} \times \underline{n} = 0$ on Γ implies that $\underline{v}$ is tangential to Γ. Hence,

$$v_r = \frac{\partial v_r}{\partial \theta} = \frac{\partial v_r}{\partial \varphi} = 0 \quad \text{ on } \Gamma. \tag{3.9}$$

Further, from (3.7), it follows

$$v_\theta = u_\varphi\, \omega_r \;\text{ and }\; v_\varphi = -u_\theta\, \omega_r, \qquad \text{on } \Gamma\, .$$

By recalling (3.2) and then using (3.7), (3.8) and (3.9), we show that the θ and the φ components of $curl\, \underline{v}$ on Γ are given by

$$[\, curl\, \underline{v}\,]_\theta = -\, \frac{1}{r}\, \frac{\partial}{\partial r}\, (r\, v_\varphi)$$

and

$$[\, curl\, \underline{v}\,]_\varphi = \, \frac{1}{r}\, \frac{\partial}{\partial r}\, (r\, v_\theta)\, ,$$

respectively.

Straightforward calculations lead to

$$[\, curl\, \underline{v}\,]_\theta = -\frac{2}{\sin^2 \theta}\, h(1) h'(1) \frac{\partial g}{\partial \varphi}\, G(\theta, \varphi) \;\text{ on } \Gamma.$$

Therefore, if $h(1) \neq 0$ (hence $h'(1) \neq 0$ by $h(1) + h'(1) = 0$) and if (3.5) is satisfied at some point $P \in \Gamma$, it follows that $[\, curl\, \underline{v}\,]_\theta \neq 0$ at P. Consequently this last quantity does not vanish in a neighborhood of P. The same arguments applied on the φ-component of $curl\, \underline{v}$ on Γ ensure that under condition (3.6) at some point P, $[\, curl\, \underline{v}\,]_\varphi \neq 0$ at P. $\square$

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